PARAMETRIC METHOD IN THE THEORY OF AN UNSTEADY BOUNDARY LAYER

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UDC 532.526.2

An approximate calculation method is described for an unsteady laminar boundary layer in an incompressible liquid. The method is based on the integration of a "universal" equation.

Several investigators have applied the parametric method of Loitsyanskii [1] to problems involving an unsteady laminar boundary layer in an incompressible liquid [2-4]. The case treated in [2, 3] was that in which the velocity at the outer boundary of the boundary layer is written as the product of two functions, one of which depends on the longitudinal coordinate while the other depends only on the time. The solution in [4], on the other hand, which holds for an arbitrary velocity function, incorporates a departure from rigor (justified only indirectly) involving the absence of a compatibility condition for the equations for the transverse scale in the boundary layer. Below we describe a parametric calculation method for an unsteady laminar boundary layer based on a "universal" equation which is valid for a broad class of velocities at the outer boundary of the boundary layer.

The stream function ψ for a plane, unsteady, laminar boundary layer in an incompressible liquid is

$$\frac{\partial^2\psi}{\partial t\,\partial y}+\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\,\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=\dot U+UU'+\nu\frac{\partial^3\psi}{\partial y^3},$$
$$\psi=\frac{\partial\psi}{\partial y}=0 \quad \text{at} \quad y=0,$$
$$\frac{\partial\psi}{\partial y}\to U(x,\ t) \quad \text{as} \quad y\to\infty, \tag{1}$$
$$\frac{\partial\psi}{\partial y}=u_1(x,\ y) \quad \text{at} \quad t=t_1,$$
$$\frac{\partial\psi}{\partial y}=u_0(t,\ y) \quad \text{at} \quad x=x_0.$$

Here and below, the prime and dot denote partial derivatives with respect to x and t, respectively. Rewriting (1) in terms of the new variables

$$x=x,\quad t=t,\quad \eta=\frac{By}{h(x,\ t)},\quad \varphi(x,\ \eta,\ t)=\frac{B\psi(x,\ y,\ t)}{U(x,\ t)\,h(x,\ t)}, \tag{2}$$

where B is a normalization constant; and h(x, t) is a scale transverse linear dimension in the boundary layer, we find ($z = h^2/\nu$):

$$B^2\frac{\partial^3\varphi}{\partial\eta^3}+Uz'\left[\varphi\frac{\partial^2\varphi}{\partial\eta^2}-\left(\frac{\partial\varphi}{\partial\eta}\right)^2+1\right]+\frac{\dot U}{U}z\left(1-\frac{\partial\varphi}{\partial\eta}\right)+$$
$$+\frac{U'z}{2}\varphi\frac{\partial^2\varphi}{\partial\eta^2}+\frac{\dot z}{2}\eta\frac{\partial^2\varphi}{\partial\eta^2}-z\frac{\partial^2\varphi}{\partial t\,\partial\eta}+Uz\left(\frac{\partial\varphi}{\partial\eta}\frac{\partial^2\varphi}{\partial\eta^2}-\frac{\partial\varphi}{\partial\eta}\frac{\partial^2\varphi}{\partial x\,\partial\eta}\right)=0, \tag{3}$$
$$\varphi=\frac{\partial\varphi}{\partial x}=0 \quad \text{at} \quad \eta=0,\quad \frac{\partial\varphi}{\partial\eta}\to 1 \quad \text{as} \quad \eta\to\infty.$$

M. I. Kalinin Leningrad Polytechnical Institute. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 27, No. 1, pp. 110-118, July, 1974. Original article submitted January 7, 1974.

The boundary conditions in terms of the variables x and t [the last lines in Eqs. (1)] are used only in the last step of the solution.

We also introduce a momentum equation for the unsteady boundary layer:

$$\frac{1}{U}\frac{\partial\delta^*}{\partial t}+\frac{\dot U}{U^2}\delta^*+\frac{\partial\delta^{**}}{\partial x}+\frac{U'}{U}(2\delta^{**}+\delta^*)=\frac{\tau_w}{\rho U^2},$$
$$\delta^*(x,\ t)=\int_0^\infty\left(1-\frac{u}{U}\right)dy,\quad \delta^{**}(x,\ t)=\int_0^\infty\frac{u}{U}\left(1-\frac{u}{U}\right)dy, \tag{4}$$

$$\tau_w(x,\ t)=\mu\left.\frac{\partial u}{\partial y}\right|_{y=0}. \tag{5}$$

We consider the dimensionless characteristic functions

$$H^*(x,\ t)=\frac{\delta^*}{h},\ H^{**}(x,\ t)=\frac{\delta^{**}}{h},\quad \zeta(x,\ t)=\frac{\tau_w h}{\mu U}; \tag{6}$$

using (5) and (2), we write these functions as

$$H^*(x,\ t)=\int_0^\infty\left(1-\frac{u}{U}\right)d\left(\frac{y}{h}\right)=\frac{1}{B}\int_0^\infty\left(1-\frac{\partial\varphi}{\partial\eta}\right)d\eta,$$

$$H^{**}(x,\ t)=\int_0^\infty\frac{u}{U}\left(1-\frac{u}{U}\right)d\left(\frac{y}{h}\right)=\frac{1}{B}\int_0^\infty\frac{\partial\varphi}{\partial\eta}\left(1-\frac{\partial\varphi}{\partial\eta}\right)d\eta, \tag{7}$$
$$\zeta(x,\ t)=\left[\frac{\partial u/U}{\partial y/h}\right]_{y=0}=B^2\left.\frac{\partial^2\varphi}{\partial\eta^2}\right|_{\eta=0}.$$

The derivatives in (4) can be written

$$\frac{\partial\delta^*}{\partial t}=H^*\dot h+h\frac{\partial H^*}{\partial t},\quad \frac{\partial\delta^{**}}{\partial x}=H^{**}h'+h\frac{\partial H^{**}}{\partial x}. \tag{8}$$

We introduce the series of parameters

$$f_{kn}=U^{k-1}\frac{\partial^{k+n}U}{\partial x^k\partial t^n}\quad z^{k+n}\,(k,\ n=0,\ 1,\ 2,\ \ldots) \tag{9}$$

and the constant parameter

$$g=\dot z=\text{const}, \tag{10}$$

where this constant can take on various values. The set of these independent parameters reflects the nature of the velocity change in the external flow and, in integral form (through z and $\dot z$), the history of the motion in the boundary layer. Using these parameters we can transform differential equation (3) to a universal form in the sense that neither the equation itself not its boundary conditions depend explicitly on U(x, t); we can write the solution of the equation in the form

$$\frac{u}{U}=\Phi(\eta,\ f_{kn},\ g), \tag{11}$$

or, for the stream function

$$\psi=\frac{Uh}{B}\varphi(\eta,\ f_{kn},\ g). \tag{12}$$

The universal equation can be derived in the following manner: transforming to the new independent variables η, f_{kn} in Eqs. (3) and (4), after first calculating the derivatives in them, we find

$$\frac{\partial\varphi}{\partial x}=\sum_{k,n=0}^{\infty}\frac{\partial\varphi}{\partial f_{kn}}\cdot\frac{\partial f_{kn}}{\partial x},$$

$$
\frac{\partial^2\varphi}{\partial x\,\partial\eta}=\sum_{k,n=0}^{\infty}\frac{\partial^2\varphi}{\partial f_{kn}\,\partial\eta}\cdot\frac{\partial f_{kn}}{\partial x},
\qquad
\frac{\partial^2\varphi}{\partial t\,\partial\eta}=\sum_{k,n=0}^{\infty}\frac{\partial^2\varphi}{\partial f_{kn}\,\partial\eta}\cdot\frac{\partial f_{kn}}{\partial t}. \tag{13}
$$

The derivatives of the parameters with respect to x and t in these equations are calculated by differentiating (9). We find

$$
\frac{\partial f_{kn}}{\partial x}=\frac{1}{Uz}\left[(k-1)f_{10}f_{kn}+(k+n)f_{kn}Uz'+f_{k+1,n}\right]=\frac{1}{Uz}D(f_{kn};\,Uz'),
$$
$$
\frac{\partial f_{kn}}{\partial t}=\frac{1}{z}\left[(k-1)f_{01}f_{kn}+(k+n)f_{kn}g+f_{k,n+1}\right]=\frac{1}{z}E(f_{kn};\,g). \tag{14}
$$

Here $D(f_{kn};\,Uz')$ and $E(f_{kn};\,g)$ denote the quantities in the corresponding brackets in the resulting equations. Using (9), (10), (13), and (14), we can rewrite (3) as

$$
B^2\frac{\partial^3\varphi}{\partial\eta^3}+f_{10}\left[\varphi\frac{\partial^2\varphi}{\partial\eta^2}-\left(\frac{\partial\varphi}{\partial\eta}\right)^2+1\right]+f_{01}\left(1-\frac{\partial\varphi}{\partial\eta}\right)+\frac{1}{2}\frac{\partial^2\varphi}{\partial\eta^2}(\varphi Uz'+\eta g)=
$$
$$
=\sum_{k,n=0}^{\infty}\left[E(f_{kn};\,g)\frac{\partial^2\varphi}{\partial f_{kn}\,\partial\eta}+D(f_{kn};\,Uz')\left(\frac{\partial\varphi}{\partial\eta}\frac{\partial^2\varphi}{\partial f_{kn}\,\partial\eta}-\frac{\partial^2\varphi}{\partial\eta^2}\cdot\frac{\partial\varphi}{\partial f_{kn}}\right)\right]. \tag{15}
$$

To cast this equation in universal form we eliminate the quantity Uz' by means of momentum equation (4), which we transform in the following manner: after the transformation to the new variables, the quantities H^*, H^{**}, and ζ in (7) become functions of the parameters f_{kn} and g alone. Then, carrying out transformations analogous to (13) in (8), along with some other straightforward manipulations, we reduce momentum equation (4) to the form

$$
\frac{Uz'}{2}H^{**}+f_{10}(2H^{**}+H^*)-\zeta+(f_{01}+g/2)H^*=-\sum_{k,n=0}^{\infty}\left[E(f_{kn};\,g)\frac{\partial H^*}{\partial f_{kn}}+D(f_{kn};\,Uz')\frac{\partial H^{**}}{\partial f_{kn}}\right]. \tag{16}
$$

From Eq. (16) we find

$$
Uz'=\frac{\zeta-f_{10}(2H^{**}+H^*)-(f_{01}+g/2)H^*-\sum\limits_{k,n=0}^{\infty}\left\{E\frac{\partial H^*}{\partial f_{kn}}+[(k-1)f_{10}f_{kn}+f_{k+1,n}]\frac{\partial H^{**}}{\partial f_{kn}}\right\}}{\frac{H^{**}}{2}+\sum\limits_{k,n=0}^{\infty}\frac{\partial H^{**}}{\partial f_{kn}}(k+n)f_{kn}}=F(f_{kn};\,g). \tag{17}
$$

Now using (17) we can rewrite (15) as

$$
B^2\frac{\partial^3\varphi}{\partial\eta^3}+f_{10}\left[\varphi\frac{\partial^2\varphi}{\partial\eta^2}-\left(\frac{\partial\varphi}{\partial\eta}\right)^2+1\right]+f_{01}\left(1-\frac{\partial\varphi}{\partial\eta}\right)+\frac{1}{2}\frac{\partial^2\varphi}{\partial\eta^2}[\varphi F(f_{kn};\,g)+\eta g]=
$$
$$
=\sum_{k,n=0}^{\infty}\left[E(f_{kn};\,g)\frac{\partial^2\varphi}{\partial f_{kn}\,\partial\eta}+D(f_{kn};\,F)\left(\frac{\partial\varphi}{\partial\eta}\frac{\partial^2\varphi}{\partial f_{kn}\,\partial\eta}-\frac{\partial^2\varphi}{\partial\eta^2}\frac{\partial\varphi}{\partial f_{kn}}\right)\right], \tag{18}
$$
$$
\varphi=\frac{\partial\varphi}{\partial\eta}=0\ \text{ at }\ \eta=0,\qquad \frac{\partial\varphi}{\partial\eta}\to 1\ \text{ as }\ \eta\to\infty,
$$
$$
\varphi=\varphi_0(\eta)\ \text{ for }\ f_{kn}=g=0\quad(k,\,n=0,\,1,\,2,\,\ldots),
$$

where $\varphi_0(\eta)$ is the Blasius solution for a steady-state boundary layer on a plate.

The resulting equation and the boundary conditions do not contain U(x, t) explicitly and in this sense are "universal." This equation is exact for a broad class of velocities U(x, t), for which z = At + C(x), where A is an arbitrary constant and C(x) is some function of the longitudinal coordinate. This latter dependence corresponds, in particular, to that, $h \sim \sqrt{\nu t}$, which is customarily used in the "exact" solution.

Equation (18) can be integrated in the m-parameter approximation once and for all. The resulting characteristic functions can be used to find an approximate solution for problems involving an arbitrary specified velocity U(x, t), expressed in terms of a sufficiently smooth function.

Before integrating we should choose some typical value as the scale value of the transverse coordinate h(x, t) in the boundary layer. In our case it is convenient to set $h = \delta^{**}$; then according to (6) we have $H^{**} = 1$ and $H^* = \delta^*/\delta^{**} = H$, and Eq. (17) becomes

$$F(f_{kn};\ g) = Uz' = 2\left[\zeta - f_{10}(2+H) - (f_{01} + g/2)H - \sum_{k,n=0}^{\infty} E(f_{kn};\ g)\frac{\partial H}{\partial f_{kn}}\right]. \tag{19}$$

In the case of a steady-state boundary layer, in which all the unsteady parameters, including g, vanish, Eq. (18) with (19) becomes the universal equation given by Loitsyanskii [1].

Equation (18) was solved in the locally three-parameter approximation; the parameters f_{10}, f_{01}, and g were retained, while the other parameters and the derivatives with respect to all parameters f_{kn} were discarded. In this case Eq. (18) is

$$B^2\frac{d^3\varphi}{d\eta^3} + \frac{(F + 2f_{10})\varphi + \eta g}{2}\frac{d^2\varphi}{d\eta^2} + f_{10}\left[1 - \left(\frac{d\varphi}{d\eta}\right)^2\right] + f_{01}\left(1 - \frac{d\varphi}{d\eta}\right) = 0,$$
$$\varphi = \frac{d\varphi}{d\eta} = 0 \quad \text{at} \quad \eta = 0,\ \frac{d\varphi}{d\eta} \to 1 \quad \text{as} \quad \eta \to \infty, \tag{20}$$
$$\varphi = \varphi_0(\eta) \quad \text{for} \quad f_{10} = f_{01} = g = 0,$$

where $\varphi_0(\eta)$ is the Blasius solution for a steady-state boundary layer on a plate. In fact, if we set $f_{10} = f_{01} = g = 0$, we find that (20) becomes

$$\frac{d^3\varphi_0}{d\eta^3} + \frac{\zeta_0}{B^2}\varphi_0\frac{d^2\varphi_0}{d\eta^2} = 0,$$

and if we set $B^2 = \zeta_0$, it becomes the familiar Blasius equation. It follows that the normalization constant must be B = 0.47.

The functional F in Eq. (20) is calculated in this approximation on the basis of Eq. (19); specifically, we use the equation

$$F(f_{10};\ f_{01};\ g) = Uz' = 2[\zeta - f_{10}(2+H) - (f_{01} + g/2)H]. \tag{21}$$

In integrating Eq. (20) on a BESM-2 computer, we use the pivotal condensation method with iterations. Figure 1a and b illustrate the results with plots of the characteristic functions ζ, F, and H as functions of the parameter f_{10} for certain values of the parameters f_{01} and g. It follows from these curves that the position of the flow-separation point is a strong function of the magnitude and sign of the unsteady-state parameter f_{01}, which is a measure of the relative local acceleration in the external flow of the boundary layer. As the positive acceleration is increased, the possibility for the occurrence of separation in the divergent region decreases; on the other hand, at large negative accelerations, separation of the boundary layer can occur at the plate and even in the convergent region. The influence of the parameter g on the characteristic functions ζ in H is slight, evident only in the separation region. As the parameter g increases, the friction increases, and the region of separation-free flow expands. In the case of an arbitrary unsteady motion of the object, with $f_{01} \neq 0$ and $g \neq 0$, the characteristics of the boundary layer are governed primarily by the parameter f_{01}.

In solving a specific problem with a specified velocity distribution U(x, t) at the outer boundary of the boundary layer, we should use momentum equation (19) in the locally three-parameter approximation. The functional $F(f_{10};\ f_{01};\ g)$ is given by Eq. (21); according to the results of the integration which is carried out once and for all in this approximation, this functional can be approximated by the function

$$F = a_1 + a_2 f_{10} + a_3 f_{01} + a_4 g, \tag{22}$$

where the coefficients a_1, a_2, a_3, and a_4 are in turn functions of the parameters. Using the expressions for the parameters, and using (22), we write (21) as

$$Uz' - a_4\dot{z} = \left(a_2 U' + a_3\frac{\dot{U}}{U}\right)z + a_1. \tag{23}$$

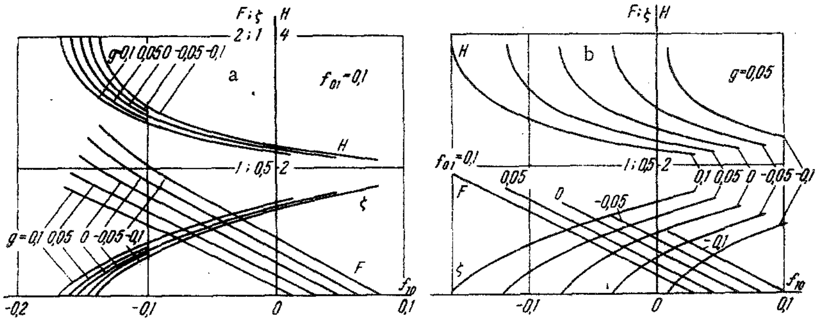

Fig. 1. a) Reduced friction coefficient and characteristic function as functions of the parameters f_{10} and g for a constant values of the parameter f_{01} = 0.1; b) as functions of the parameters f_{10} and f_{01} for a constant value g = 0.05.

According to the last lines in system (1), the boundary and initial conditions for Eq. (23) are

$$\begin{aligned} z &= z_0(t) \quad \text{at} \quad x = x_0, \\ z &= z_1(x) \quad \text{at} \quad t = t_1. \end{aligned} \tag{24}$$

Equation (23) is nonlinear, so that, in general, it must be solved approximately. However, if the values of the parameters are small, we can use Eq. (22) to approximate the unsteady functional F by a linear function with the following coefficients:

$$a_1 = 0.44, \quad a_2 = -5.35, \quad a_3 = -1.65, \quad a_4 = -2.1.$$

The coefficients are calculated as the average values over the parameter ranges, $-0.1 \le f_{01} \le 0.1$ and $-0.2 \le g \le 0.2$ and over the range of f_{10} from its value at the separation point to its value at F = 0. The deviation of the coefficients from their average values in these parameter ranges does not exceed 3-12%. Then Eq. (23) becomes linear, and it becomes a simple matter to solve this equation, determine z(x, t) and $\dot{z}$(x, t), and find the parameter values

$$f_{10}(x, t) = \dot{U}'z, \quad f_{01}(x, t) = \frac{\dot{U}}{U} z, \quad g(x, t) = \dot{z}$$

corresponding to the specified velocity U(x, t). Then the reduced friction ζ can be found by means of tables, graphs, or the approximating function $\zeta = \zeta(f_{10}; f_{01}; g)$.

We note that refinement of the solution of this problem involves solving a nonlinear equation; furthermore, the refinement requires a more rigorous determination of the characteristic functions. This can be achieved by integrating Eq. (18), retaining the derivatives with respect to the parameters f_{10} and f_{01}, and then, if the computer memory permits, retaining the derivatives with respect to the subsequent parameters in the series f_{kn}. However, satisfactory results can be achieved on the basis of the linearized equation and by taking into account only the first parameters in the local approximation, as in the case of a steady-state boundary layer [1]. The following concrete examples verify this assertion.

We consider the flow of a viscous liquid around an infinite plane, which is abruptly put into motion at a velocity U (the Rayleigh problem). For this case Eq. (27) becomes

$$2.1 \frac{dz}{dt} = 0.44,$$

from which we find the parameters of this problem to be $f_{10} = f_{01} = 0$ and $g \simeq 0.21$. Now, using available tables or graphs, and converting to $h = \delta^*$, we can easily determine the reduced friction: $\zeta = \partial(u/U)/\partial(y/\delta^*)|_{y=0}$ = 0.637. For comparison, Table 1 shows the results of the following solutions: the exact solution [5]; the approximate single-parameter solutions of Rozin [6], in which the exact Hartree solutions (for a steady-state boundary layer) or Watson solutions (the first approximation of the problem of the development of the boundary layer) are used to calculate the characteristic functions; the approximate solution of Yang [7]; and, finally, the approximate solution of Struminskii [8], who generalized the Pohlhausen method to the case of unsteady motion.

TABLE 1. Values of the Reduced Friction on an Infinite Surface which Is Abruptly Put into Motion According to Various Calculation Methods

Solution method	ζ
Exact	0,637
Rozin method with the Hartree solution	0,5715
Rozin method with the Watson solution	0,637
Yang method	0,641
Struminskii method (generalization of the Pohlhausen method)	0,600
Method of the present paper	0,637

We note that this example corresponds to the case of large local accelerations in the boundary layer, so that the largest error comes from those methods in which families of steady-state velocity profiles are used to determine the characteristic functions.

In general, in the problem of unsteady flow around a cylinder, nonlinear equation (23) should be solved, since in this problem the dependence of the coefficients on the parameters in (22) turns out to be important. A satisfactory result was found in calculating the time of separation at a cylinder abruptly put into motion by expanding the characteristic functions in series in terms of the parameters and then discarding the nonlinear terms. This result is $t_s = 0.26\ r/V_0$; the result of the exact solution is $t_s = 0.32\ r/V_0$.

NOTATION

- x, y are the longitudinal and transverse coordinates in the boundary layer;
- t is the time;
- η is the dimensionless transverse coordinate;
- U is the velocity at the outer boundary of the boundary layer;
- ψ is the stream function;
- φ is the dimensionless stream function;
- u, v are the projections of the velocity in the boundary layer onto the x and y axes, respectively;
- ρ is the liquid density;
- μ, ν are the dynamic and kinematic viscosity coefficients, respectively;
- h is the scale transverse coordinate in the boundary layer;
- $z = h^2/\nu$;
- F, H, H^*, H^{**} are the characteristic functions;
- δ^* is the displacement length;
- δ^{**} is the momentum-loss length;
- τ_w is the surface-friction stress
- ζ is the reduced friction coefficient;
- B is the normalization factor;
- f_{kn}, g are the dimensionless parameters;
- r is the cylinder radius.

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